Name of the Course: Linear Algebra

Syllabus:

Unit I

Symmetric and Skew symmetric matrices - Hermitian and Skew Hermitian Matrices – Orthogonal and Unitary matrices - Rank of matrix - Eigen values and Eigen vectors of Linear operators – Cayley Hamilton theorem - Solutions of Homogeneous linear equations – Solutions of non homogenuous linear equations.

Section 1.1 Symmetric and Skew symmetric matrices

Definition 1.1.1: Symmetric Matrix

A Matrix A is symmetric if and only if the condition $A = A^T$. A is symmetric if and only if $a_{ij} = a_{ji.}$ where a_{ij} represents an element in the i^th^ and j^th^ column of the matrix A.

Example: $A = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 5 & 6 \\ 2 & 6 & 3 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 5 & 6 \\ 2 & 6 & 3 \end{pmatrix}$

Definition 1.1.2 : Skew - Symmetric Matrix

A Matrix A is skew - symmetric if and only if the condition $A = -A^T$. A is skew - symmetric if and only if $a_{ij} = -a_{ji.}$ where a_{ij} represents an element in the i^th^ and j^th^ column of the matrix A.

Example: $A = \begin{pmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{pmatrix}$ $A^T = \begin{pmatrix} 0 & -4 & -5 \\ 4 & 0 & -6 \\ 5 & 6 & 0 \end{pmatrix}$

Theorem 1.1.3:

If a matrix A is symmetric and invertible , then A^{-1} is also symmetric.

Proof: It is given that A is symmetric. Therefore, by definition, we get $A = A^T$

Since A is invertible A^{-1} exists and we know that $(A^{-1})^T = (A^T)^{-1}$

Now, $A = A^T \implies A^{-1} = (A^T)^{-1} = (A^{-1})^T$. Therefore A^{-1} is symmetric.

Example 1.1.4: The product of the two symmetric matrices need not be symmetric.

Proof. Let $A = \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$. We can easily verify that $A = A^T$ and $B = B^T$.

Now we compute $AB = \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1+4 & 1+20 \\ 4+2 & 4+10 \end{pmatrix} = \begin{pmatrix} 5 & 21 \\ 6 & 14 \end{pmatrix}$

$(AB)^T = \begin{pmatrix} 5 & 6 \\ 21 & 14 \end{pmatrix}$. Since $AB \neq (AB)^T$. The product matrix AB is not symmetric.

Theorem 1.1.5:

If A is any symmetric matrix prove that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric.

Proof: It is given that A is symmetric matrix. Therefore by definition we get, $A = A^T$.

Now $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$. $\therefore A + A^T$ is symmetric matrix.

Similarly, $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$. $\therefore A - A^T$ is skew symmetric matrix. Hence the proof.

Theorem 1.1.6

(i) Show that Any square matrix can be explained as the sum of the symmetric and skew symmetric matrix. (ii) Also prove that this representation is unique.

Proof:

Let $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

$A + A^T$ is symmetric implies that $\frac{A+A^T}{2}$ is also symmetric. Similarly, $A - A^T$ is skew symmetric implies that $\frac{A-A^T}{2}$ is also skew symmetric. Therefore any square matrix can be explained as the sum of the symmetric and skew symmetric matrix.

(ii) we have to prove that this representation is unique. If $A = A_1 + A_2$ and $B = B_1 + B_2$ where A_1 and B_1 are symmetric matrices and A_2 and B_2 are skew symmetric matrices. We have to show that $A = B$. i.e $A_1 + A_2 = B_1 + B_2$

i.e $A_1 - B_1 = B_2 - A_2 \implies$ LHS is symmetric matrix and RHS is Skew symmetric. This is possible only when $A_1 - B_1 = 0$ and $B_2 - A_2 = 0$. i.e $A_1 = B_1$ and $B_2 = A_2$. Hence, this representation is unique.

Problem 1.1.7:

Express the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$ as the sum of the symmetric and skew symmetric matrix.

Solution:

We know that $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

$$\frac{A+A^T}{2} = \frac{1}{2}\left[\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 2 & 5 & 0 \\ 5 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

$$\frac{A-A^T}{2} = \frac{1}{2}\left[\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$\frac{A+A^T}{2} + \frac{A-A^T}{2} = \frac{1}{2}\begin{pmatrix} 2 & 5 & 0 \\ 5 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix} 2 & 5 & 0 \\ 5 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 4 & 2 \\ 6 & 0 & 2 \\ -2 & 4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

Hence the matrix A can be expressed as the sum of the symmetric and skew symmetric matrix.

$$\text{i.e } A = \begin{pmatrix} 1 & 5/2 & 0 \\ 5/2 & 0 & 3/2 \\ 0 & 3/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & -1/2 \\ -1 & 1/2 & 0 \end{pmatrix}$$

Problem 1.1.8:

Express the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ as the sum of the symmetric and skew symmetric matrix.

Solution:

We know that $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

$$\frac{A+A^T}{2} = \frac{1}{2}\left[\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 2 & 3 & 2 \\ 3 & 4 & 7 \\ 2 & 7 & 18 \end{pmatrix}$$

$$\frac{A-A^T}{2} = \frac{1}{2}\left[\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\frac{A+A^T}{2} + \frac{A-A^T}{2} = \frac{1}{2}\begin{pmatrix} 2 & 3 & 2 \\ 3 & 4 & 7 \\ 2 & 7 & 18 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix} 2 & 3 & 2 \\ 3 & 4 & 7 \\ 2 & 7 & 18 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 2 & 4 \\ 4 & 4 & 6 \\ 2 & 8 & 18 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

Hence the matrix A can be expressed as the sum of the symmetric and skew symmetric matrix.

$$\text{i.e } A = \begin{pmatrix} 1 & 3/2 & 1 \\ 3/2 & 2 & 7/2 \\ 1 & 7/2 & 9 \end{pmatrix} + \begin{pmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 \end{pmatrix}$$

Theorem 1.1.9

If A and B are skew symmetric matrices then prove that (i) A+B is skew symmetric matrix (ii) A^{2n} is symmetric. (iii) A^{2n+1} is skew symmetric matrix where $n \in Z$.

Proof: It is given that A and B are skew symmetric matrices. Therefore $A = -A^T$ and $B = -B^T$. Now we rove

(i) $A + B = -A^T + (-B^T) = -(A^T + B^T) = -(A + B)^T$. Therefore $A + B$ is skew symmetric matrix.

(ii) Consider $A^{2n} = (-A^T)^{2n}$ $\because$ A is skew symmetric matrix.

$$= (-1)^{2n} (A^T)^{2n} = (A^{2n})^T$$

Which implies that A^{2n} is symmetric matrix.

(iii) Consider $A^{2n+1} = (-A^T)^{2n+1}$ $\because$ A is skew symmetric matrix.

$$= (-1)^{2n+1} (A^T)^{2n+1} = -(A^{2n+1})^T$$

Which implies that A^{2n+1} is skew symmetric matrix.

Theorem 1.1.10

If A and B are invertible, symmetric and commuting matrices then **show** the following (i) $A^{-1}B$ is symmetric (ii) AB^{-1} is symmetric (iii) $A^{-1}B^{-1}$ is symmetric

Proof: It is given that A and B are symmetric, invertible and commuting matrices.

Therefore from the given conditions we have

$A = A^T$ and $B = B^T$ .... (1) A^{-1}, B^{-1} exists ...... (2) $AB = BA$ .... (3) holds.

(i) In (3) Pre multiply by A^{-1} and Post multiply by B^{-1} we gat,

$$A^{-1}AB\,B^{-1} = A^{-1}BA\,B^{-1}$$

$$I = A^{-1}BA\,B^{-1}$$

$$\Rightarrow \left(A^{-1}B\right)^{-1} = A\,B^{-1}$$

$$\Rightarrow B^{-1}\left(A^{-1}\right)^{-1} = A\,B^{-1}$$

$$\Rightarrow B^{-1}A = A\,B^{-1} \quad \text{...... (4)}$$

Similarly, In (3) Pre multiply by B^{-1} and Post multiply by A^{-1} we gat,

$$B^{-1}ABA^{-1} = B^{-1}BAA^{-1}$$

$$B^{-1}ABA^{-1} = I$$

$$\Rightarrow BA^{-1} = \left(B^{-1}A\right)^{-1}$$

$\Rightarrow \; BA^{-1} = \; A^{-1}B$ ...... (5)

(i) $(A^{-1}B)^T = (BA^{-1})^T$ $\because$ $From$ equation (5)

$= B^T(A^{-1})^T$

$= B^T(A^T)^{-1}$

$= B\,(A)^{-1}$ $\because$ $From$ equation (1)

$= BA^{-1}$

$= A^{-1}B$ $\because$ $From$ equation (5)

Hence $A^{-1}B$ is symmetric.

(ii) $(AB^{-1}\;)^T = (B^{-1}\,A)^T$ $\because$ $From$ equation (4)

$= A^T(B^{-1})^T$

$= A^T(B^T)^{-1}$

$= A\,(B)^{-1}$ $\because$ $From$ equation (1)

$= AB^{-1}$

Hence AB^{-1} is symmetric.

(i) $(A^{-1}B^{-1}\;)^T = (B^{-1}\,)^T\,(A^{-1}\,)^T$ $\because$ $From$ equation (4)

$= (B^T\,)^{-1}\,(A^T\,)^{-1}$

$= \; B^{-1}A^{-1}$ $\because$ $From$ equation (1)

$= \; (AB)^{-1}$

$= (BA)^{-1}$ $\because$ $From$ equation (3)

$= \; A^{-1}B^{-1}$

Hence $A^{-1}B^{-1}$ is symmetric.